References:

Last time: FD boundary conditions and introduce 1-D parabolic equation.

Reading assignment: Franke and Swenson, “A brief tutorial on the fast field program (FFP) . . .”

History of PE in acoustics (abbreviated):
- First came: “Fourier split step PE” — late 70s to early 80s
  - some situations
- Next came: “Implicit finite differences PE” — mid to late 80s
  - many situations, can be slow
- Today: “Green’s function PE” (Gilbert and Di) — 90s
  - more general situations
  - like Fourier split step but can be faster
  - still gaining acceptance

Angle limitations on the PE

The PE we derived previously, in which we threw out the \( \frac{\partial^2 \psi}{\partial z^2} \) term, is only valid for angles within \( \pm 20^\circ \) of horizontal. Thus, it is called a narrow angle PE.
For years people have worked on improved versions. The usual approach is to factor the Helmholtz equation as

\[
\left( \frac{\partial}{\partial r} + i\sqrt{Q} \right) \left( \frac{\partial}{\partial r} - i\sqrt{Q} \right) p = 0
\]

where \( Q \) is an operator. The forward going part is

\[
\frac{\partial p}{\partial r} = i\sqrt{Q}p
\]

The essence of a wide angle PE is a very accurate approximation of \( \sqrt{Q} \).

If we write \( \sqrt{Q} = k_0\sqrt{1+q} \), there are many available ways to approximate \( \sqrt{1+q} \). One of the most accurate is by several terms of a Padé approximation. (See pp. 347-357 of Jensen, et al.) A two term Padé approximation yields a PE sufficiently accurate for angles \( \pm 55^\circ \) of horizontal. Five terms is accurate for angles \( \pm 75^\circ \).

See pp. 38–41 of West, Gilbert, and Sack for an example implementation.
Wavenumber integration methods:

- major competitor to the PE over the years
- assumes a vertically stratified medium, $c = c(z)$
- SAFARI is one such code used in underwater acoustics, another is

The fast field program (FFP)

Like SAFARI and other wavenumber integration programs, the FFP is based on a Green’s function solution of the Helmholtz equation.

Assume a point source at $(0, 0, z_s)$:

$$\nabla^2 p + k^2 p = -4\pi \delta(x, y, z - z_s)$$

where $p = p(\vec{x})$ is the complex amplitude for a particular frequency $\omega$ and

$$k = \frac{\omega}{c(z)}.$$

Note that $c = c(z)$ only! The speed of sound cannot depend on range $r$.

Rewrite in cylindrical coordinates $(r, \phi, z)$ assuming no variations w.r.t. $\phi$:

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} + \frac{\partial^2 p}{\partial z^2} + k^2 p = -\frac{2}{r} \delta(r) \delta(z - z_s)$$
Transform domain solution
Solve in the transform domain. Since in cylindrical coordinates, use zeroth order Hankel Transform:

- Forward transform:

\[ \hat{p}(K, z) = \int_0^\infty p(r, z)J_0(Kr)rdr \]

- Inverse transform:

\[ p(r, z) = \int_0^\infty \hat{p}(K, z)J_0(Kr)KdK \]

where \( \hat{p} \) is the Hankel transform of \( p \).

[see handout on Hankel transforms]

If you do the Hankel transforms:

\[ \frac{d^2\hat{p}}{dz^2} + [k^2(z) - K^2] \hat{p} = -2\delta(z - z_s) \]
Solving the transformed equation

\[
\frac{d^2 \hat{p}}{dz^2} + [k^2(z) - K^2] \hat{p} = -2\delta(z - z_s)
\]

is a 1-D inhomogeneous Helmholtz equation which should be solved in the \( z \) direction. (The \( r \) direction solution will be given by the inverse Hankel transform once the above equation is solved.)

To solve, Franke and Swenson break it up into 2 first order equations. Recall Euler’s equation (\( e^{j\omega t} \) notation assumed) in the \( z \) direction:

\[
j\omega \rho_0 u_z + \frac{dp}{dz} = 0 ,
\]

and taking Hankel transforms,

\[
j\omega \rho_0 \hat{u}_z + \frac{d\hat{p}}{dz} = 0 .
\]

If we take the derivative of both sides of this equation w.r.t \( z \) we have

\[
\frac{d\hat{u}_z}{dz} = \frac{j}{\omega \rho_0} \frac{d^2\hat{p}}{dz^2} .
\]
Thus, we can rewrite

\[ \frac{d^2 \hat{p}}{dz^2} + [k^2(z) - K^2] \hat{p} = -2\delta(z - z_s) \]

as

\[ \frac{d \hat{p}}{dz} = -j\omega \rho_0 \hat{u}_z \]

and

\[ \frac{d \hat{u}_z}{dz} = \frac{j}{\omega \rho_0} \left( -[k^2(z) - K^2] \hat{p} - 2\delta(z - z_s) \right) \cdot \]

Swenson and Franke then solve these two coupled equations in an ANALOGY with Inhomogeneous Electrical Transmission Lines:

- voltage \( V \) is analogous to pressure \( \hat{p}(K, z) \)
- current \( I \) is analogous to particle velocity \( \hat{u}_z(K, z) \)

Once \( \hat{p}(K, z) \) is known, the inverse Hankel transform gives \( p(r, z) \).
Approximating the inverse transform

The inverse transform is

\[ p(r, z) = \int_0^\infty \hat{p}(K, z) J_0(Kr) K dK \]

where we integrate over wavenumber. This is why the FFP is called a wavenumber integration method.

Doing the integration directly can be very slow. Instead, we make some approximations in the above equation. Recall

\[ J_0(Kr) = \frac{1}{2} \left[ H_0^{(1)}(Kr) + H_0^{(2)}(Kr) \right] , \]

and if we only have outgoing waves

\[ J_0(Kr) = \frac{1}{2} H_0^{(2)}(Kr) . \]

In the far field where \( Kr \gg 1 \), one can approximate

\[ H_0^{(2)} \approx \sqrt{\frac{2}{\pi K}} \frac{e^{-j(Kr-\pi/4)}}{\sqrt{r}} \]
Hence, an approximate version of the inverse transform is

\[ p(r, z) \approx \frac{1 + j}{\sqrt{2\pi r}} \int_0^\infty \hat{p}(K, z)e^{-jKr} \sqrt{K} dK \]

This integral is approximated by

\[ p(r, z) \approx \frac{1 + j}{\sqrt{2\pi r}} \int_0^{\text{BIG}} \hat{p}(K, z)e^{-jKr} \sqrt{K} dK \]

which has a finite integration range.

**Where the FFP gets its name**

Integrate using a fast Fourier transform (FFT) after rearranging:

\[ p(r, z) \approx \frac{1 + j}{\sqrt{2\pi r}} \int_0^{\text{BIG}} \left( \hat{p}(K, z)\sqrt{K} \right) e^{-jKr} dK \]

In addition a small amount of damping is added to avoid any possible poles along the \( K \) axis. Since the inverse Fourier transform of \( \hat{f}(K - j\alpha) \) is \( f(r)e^{-\alpha r} \), we use

\[ p(r, z) \approx e^{\alpha r} \frac{1 + j}{\sqrt{2\pi r}} \int_0^{\text{BIG}} \left( \hat{p}(K - j\alpha, z)\sqrt{K - j\alpha} \right) e^{-jKr} dK \]
where $\alpha$ is called an artificial attenuation. This attenuation is particularly important for avoiding a branch point which occurs at $K = \omega/c_{\text{top}}$ where $c_{\text{top}}$ is the sound speed of the top layer. At that wavenumber $\hat{p}(K,z) \to \infty$. (See pp. 237-238 of Jensen, et al.)

**Caveats**
- Because of the artificial attenuation, the FFP is not good for long ranges.
- $c(z)$ only!
- The ground impedance cannot vary with range.

**Benefits**
- FFP is probably the most accurate method for short ranges. (Much more accurate than a PE for large angles.)
- For a run with a source at a particular height, the FFP gives you solutions at all ranges in one solve.
Summary of Propagation Methods

Ray Theory: [Numerically step rays by carefully monitoring the direction cosines of the rays, depending on $c(\vec{x})$. See Chap. 3 of Jensen et al. See also Chap 8 of Pierce and pp. 117–120 of Kinsler, et al.]

- almost any environment is OK.
- very flexible, but sensitive to environment.
- bad: low freq.
- bad: often need to calculate many, many rays.
- must do special things when encounter caustics and shadow zones.

Normal Mode Theory: [Numerically calculate orthogonal modes of underwater “duct” for complicated $c(z)$. See Chap. 5 of Jensen et al. See also p. 430 of Kinsler, et al. for analytical version.]

- assumes $c(z)$ only.
- heavily dependent on the boundary conditions to find the modes.
- must find roots of characteristic equation accurately and uniquely.
- difficulties for deep water?
FFP (wavenumber integration):
— assumes \( c(z) \) only.
— possibly the best method for moderately short ranges for fixed freq.
— each run gives you \( p \) at one height and one \( \omega \) for ALL ranges
— inefficient for the calculation of pulses.

PE:
— almost any environment is OK.
— typical finite difference or Tappert formulation can be slow, really slow for high frequencies and long ranges.
— what value of reference wavenumber \( k_0 \)?
— what is a good starting field?
— only good within certain angles
  → problems with moderately short ranges (angle violations)

Next time: Start working toward finite elements.